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ORIGINAL PAPER

Information theoretic spreading measures of orthogonal functions

D. Nath

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Abstract We calculate information theoretic spreading measures of *orthogonal functions* associated with solutions of quantum mechanical isospectral potentials. In particular, Shannon, Renyi and Fisher lengths have been evaluated for potentials isospectral to the linear harmonic oscillator and the symmetric Rosen-Morse potentials. We have also compared the behaviour of different lengths for the *orthogonal functions* and the associated orthogonal polynomials.

1 Introduction

In recent years there have been growing interest in studying families of non trivial potentials which are isospectral partners [1,2] of well known solvable potentials like the harmonic oscillator [3,4]. Similar ideas have also been examined in the context of supersymmetric quantum mechanics (SUSYQM) [5–10]. It is interesting to note that while the solutions for most standard solvable quantum mechanical potentials are given in terms of classical orthogonal polynomials, the solutions of the corresponding isospectral partners (depending on one or more parameters) are generally given in terms of *orthogonal functions* asbuilt up from classical orthogonal polynomials [11–13].

In this context it may be noted that various information theoretic measures of uncertainty have been studied in great detail for the classical orthogonal polynomials which furnish solutions of such well known problems like the Harmonic oscillator, Coulomb potential, Morse potential, Pöschl-Teller potential etc [14]. Although isospectral deformation of the above mentioned potentials have been studied by a number of authors in the context of Darboux theroem or supersymmetric quantum mechanics, barring a few

D. Nath (✉)
Department of Mathematics, Vivekananda College, 269 D.H. Road, Kolkata, 700063, India
e-mail: debrajn@gmail.com

Table 1 Fisher lengths, Renyi lengths and Shannon lengths of the state $\widehat{\psi}_n^-(x, \mu, \lambda)$ in Eq. (24) for $\mu = 1$ and $\lambda = 1$

n	$(\widehat{\delta x}_H)_n^-(\lambda, \mu)$	$\widehat{\mathfrak{L}}_{Hn}^R(\lambda, \mu)$	$\widehat{N}_H[\rho_n](\lambda, \mu)$
0	0.692098	3.51462	2.88259
1	0.413237	4.49039	3.79337
2	0.315388	5.21221	4.48006

Table 2 Fisher lengths, Renyi lengths and Shannon lengths of the state $\widehat{\psi}_n^-(x, \gamma, \lambda)$ in Eq. (31) for $\gamma = 4$ and $\lambda = 1$

n	$(\widehat{\delta x}_R)_n^-(\lambda, \gamma)$	$\widehat{\mathfrak{L}}_{Rn}^R(\lambda, \gamma)$	$\widehat{N}_R[\rho_n](\lambda, \gamma)$
0	2.70648	1.93334	1.53992
1	4.09497	2.86675	2.38479
2	4.62147	4.30161	3.46009

exceptions [15] they have not been examined much from the point of view of information theoretic uncertainty. Here our objective is to compute the spreading measures for the *orthogonal functions* which form solutions of isospectrally deformed potentials and compare them with the same quantities for the associated classical orthogonal polynomials.

It may be noted that the most familiar spreading measure is the root mean-square or standard deviation

$$\Delta x = \sqrt{(\langle x^2\rangle - \langle x\rangle^2)} \tag{1}$$

where the expectation value of a function $f(x)$ is defined by

$$\langle f(x)\rangle = \int f(x)\rho_n(x)dx \tag{2}$$

where $\rho_n(x)$ is a probability distribution function. However this measure is not always a particularly suitable measure of spreading and consequently we shall consider various other information theoretic measures of spreading like the Shannon, Renyi and Fisher lengths (Tables 1, 2).

Let us note that the Fisher information, the Renyi entropy of order q and the Shannon entropy are given respectively by [16–18]

$$F[\rho_n] = \left\langle \left[\frac{d}{dx}\ln\rho_n(x)\right]^2\right\rangle = \int_{-\infty}^{\infty} \frac{1}{\rho_n(x)}\left(\frac{d}{dx}\rho_n(x)\right)^2 dx \tag{3}$$

$$R_q[\rho_n] = \frac{1}{1-q}\ln\langle[\rho_n(x)]^{q-1}\rangle \tag{4}$$

$$S[\rho_n] = -\int_{-\infty}^{\infty} \rho_n(x)\ln\rho_n(x)dx \tag{5}$$

Since the Fisher, Renyi and Shannon entropies corresponding to a given density $\rho_n(x)$ have particular units, which are different from that of the variable x, it is much more useful to use the related information-theoretic lengths associated with these measures. Thus we consider the Fisher, Renyi and Shannon length defined respectively by [19,20]

$$(\delta x)_n = \frac{1}{\sqrt{F[\rho_n(x)]}} \tag{6}$$

$$\mathfrak{L}_n^R = exp\{R_q[\rho_n(x)]\} \tag{7}$$

$$N[\rho_n] = exp\{S[\rho_n(x)]\} \tag{8}$$

These three quantities together with the standard deviation will be referred as the direct spreading measures of the density $\rho_n(x)$ because they share the following properties: linear scaling under adequate boundary conditions, same units as the variable, and vanishing when the density tends to delta density. Moreover, they have an associated uncertainty properties and fulfil the inequalities [19]

$$\begin{aligned} (\delta x)_n &\leq (\Delta x)_n \\ N[\rho_n(x)] &\leq \sqrt{2\pi e}(\Delta x)_n \end{aligned} \tag{9}$$

2 SUSY QM and construction of isospectral Hamiltonian

We note that a one dimensional SUSY Quantum mechanical model consists of a pair of Hamiltonians $H^{\pm}$ of the form [5]

$$H^{\pm}(x) = A^{\pm}A^{\mp} = -\frac{d^2}{dx^2} + V_{\pm} \tag{10}$$

where

$$\begin{aligned} A^{\pm} &= \pm\frac{d}{dx} + W(x) \\ V_{\pm}(x) &= W^2(x) \pm W'(x) \end{aligned} \tag{11}$$

.

In this case the relation between the energies and the eigenstates of the Hamiltonian $H^{\pm}$ are given by

$$\begin{aligned} E_0^- &= 0,\ E_{n+1}^- = E_n^+ \\ \psi_n^+(x) &= \frac{1}{\sqrt{E_{n+1}^-}}A^+\psi_{n+1}^-(x),\ \psi_{n+1}^-(x) = \frac{1}{\sqrt{E_n^+}}A^-\psi_n^+(x) \end{aligned} \tag{12}$$

We assume that the ground state belongs to the Hamiltonian H^- and the corresponding ground state is $\psi_0^-(x) = C_0^- exp[-\int W(x)dx]$, where C_0^- is the normalization constant.

To construct non trivial isospectral potentials let us now consider another superpotential $\widehat{W}(x)$ such that

$$\widehat{W}^2(x) + \widehat{W}'(x) = W^2(x) + W'(x) \tag{13}$$

Clearly one solution of the above equation is $\widehat{W}(x) = W(x)$ while the other solution is given by [21]

$$\widehat{W}(x) = W(x) + \frac{e^{-2\int W(x)dx}}{\lambda + e^{-2\int W(x)dx}} \tag{14}$$

where λ is an integration constant which has to be so chosen that $\widehat{W}$ is non singular. It may be noted that $\widehat{V}_-(x) = \widehat{W}^2(x) - \widehat{W}'(x)$ is a new potential which is isospectral to $V_-(x)$. The normalized ground state wave function and the excited state wave functions corresponding to the potential $\widehat{V}_-(x,\lambda)$ are given by [5]

$$\begin{aligned}
\widehat{\psi}_0^-(x,\lambda) &= \frac{\sqrt{\lambda(\lambda+1)}\psi_0^-(x)}{\lambda + \mathcal{I}(x)} = \sqrt{\omega(x)}\Phi_0(x) \\
\widehat{\psi}_{n+1}^-(x,\lambda) &= \psi_{n+1}^-(x) + \frac{1}{E_{n+1}^-}\left(\frac{\mathcal{I}'(x)}{\lambda+\mathcal{I}(x)}\right)\left(\frac{d}{dx} + W(x)\right)\psi_{n+1}^-(x) \\
&= \frac{1}{\lambda+\mathcal{I}(x)}\left[(\lambda+\mathcal{I}(x))\psi_{n+1}^-(x) + \frac{\mathcal{I}'(x)}{E_{n+1}^-}\left(\frac{d}{dx} + W(x)\right)\psi_{n+1}^-(x)\right] \\
&= \sqrt{\omega(x)}\Phi_{n+1}(x)
\end{aligned} \tag{15}$$

where $\lambda > 0$ or $\lambda < -1$ and

$$\begin{aligned}
\mathcal{I}(x) &= \int_{-\infty}^{x} \psi_0^{-2}(t)dt \\
\omega(x) &= \frac{1}{(\lambda + I(x))^2} \\
\Phi_0(x) &= \sqrt{\lambda(\lambda+1)}\psi_0^-(x) \\
\Phi_{n+1}(x) &= [\lambda + \mathcal{I}(x)]\psi_{n+1}^-(x) + \frac{\mathcal{I}'(x)}{E_{n+1}^-}\left(\frac{d}{dx} + W(x)\right)\psi_{n+1}^-(x)
\end{aligned} \tag{16}$$

It is not difficult to observe that the wave functions in (15) are given in terms of the functions $\Phi_0(x)$ and $\Phi_{n+1}(x)$ which are in general not polynomials orthogonals multiplied by a weight function. However, they are orthogonal with respect to the weight function $\omega(x)$ i.e,

$$\int_{-\infty}^{\infty} \Phi_m(x)\Phi_n(x)\omega(x)dx = \delta_{mn} \tag{17}$$

This relation actually follows from the standard orthonormality relation in quantum mechanics

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = \delta_{mn} \tag{18}$$

We would like to point out that when $\lambda \to \infty$ the wave functions $\widehat{\psi}_n^-(x, \lambda) \to \psi_n^-(x)$ and the orthogonal functions $\Phi_n(x)$ reduces to some known orthogonal polynomials depending on the superpotential $W(x)$ and the weight function $\omega(x)$ reduces to the corresponding weight function of the orthogonal polynomials. In the next section we shall construct the isospectral partners of the linear harmonic oscillator and the Rosen-Morse potential.

3 Examples

3.1 The linear Hermonic oscillator

The first example we shall consider is characterized by a superpotential $W(x) = \mu x$. In this case $V_+(x)$ is given by

$$V_+(x) = \mu^2 x^2 \pm \mu \tag{19}$$

The above potentials are standard linear harmonic oscillator potential (shifted in the energy scale). The energy eigenvalues and eigenfunctions of $V_+(x)$ are given by

$$\begin{aligned} E_n^+ &= 2(n+1)\mu, \quad n = 0, 1, 2, \ldots.. \\ \psi_n^+(x) &= \sqrt{\frac{\sqrt{\mu}}{2^n (n)! \sqrt{\pi}}} e^{-\frac{\mu x^2}{2}} H_n(\sqrt{\mu} x) \end{aligned} \tag{20}$$

while those of the partner potential $V_-(x)$ are given by

$$\begin{aligned} E_n^- &= 2n\mu, \quad n = 0, 1, 2, \ldots.. \\ \psi_n^-(x) &= \sqrt{\frac{\sqrt{\mu}}{2^n n! \sqrt{\pi}}} e^{-\frac{\mu x^2}{2}} H_n(\sqrt{\mu} x) \end{aligned} \tag{21}$$

where H_n denotes the Hermite polynomial [22].

The eigenvalues and eigenfunction [9] for the isospectral Hamiltonian $\widehat{V}_-(x)$ are given by

$$\begin{aligned}
\widehat{E}_0^- &= 0\\
\widehat{E}_{n+1}^- &= E_{n+1}^-\\
\widehat{\psi}_0^-(x,\mu,\lambda) &= \frac{\sqrt{\lambda(\lambda+1)}(\frac{\mu}{\pi})^{\frac{1}{4}}e^{-\frac{\mu x^2}{2}}}{\left(\lambda+\sqrt{\frac{\mu}{\pi}}\int_{-\infty}^{x}e^{-\mu t^2}dt\right)}\\
\widehat{\psi}_{n+1}^-(x,\mu,\lambda) &= \psi_{n+1}^-(x)+\frac{1}{E_{n+1}^-}\left(\frac{\mathcal{I}'(x)}{\lambda+\mathcal{I}(x)}\right)\left(\frac{d}{dx}+W(x)\right)\psi_{n+1}^-(x)\\
&= \frac{1}{\lambda+\mathcal{I}(x)}\left((\lambda+\mathcal{I}(x))\,\psi_{n+1}^-(x)+\frac{\mathcal{I}'(x)}{\sqrt{E_{n+1}^-}}\psi_n^-(x)\right)\\
&= \sqrt{\frac{\sqrt{\mu}}{2^{n+1}(n+1)!\sqrt{\pi}}}\frac{e^{-\frac{\mu x^2}{2}}}{(\lambda+\mathcal{I}(x))}\\
&\quad\times\left((\lambda+\mathcal{I}(x))\,H_{n+1}(\sqrt{\mu}x)+\frac{\mathcal{I}'(x)}{\sqrt{\mu}}H_n(\sqrt{\mu}x)\right)
\end{aligned}\tag{22}$$

where

$$\mathcal{I}(x)=\sqrt{\frac{\mu}{\pi}}\int_{-\infty}^{x}e^{-\mu t^2}dt \tag{23}$$

Next, for the sake of simplicity we choose $\mu=1$. In this case

$$\begin{aligned}
\widehat{\psi}_0^-(x,\lambda) &= \frac{\sqrt{\lambda(\lambda+1)}e^{-\frac{x^2}{2}}}{\sqrt{\sqrt{\pi}}\,(\lambda+0.5+0.5Erf[x])}\\
\widehat{\psi}_1^-(x,\lambda) &= \sqrt{\frac{1}{2\sqrt{\pi}}}\frac{e^{-\frac{x^2}{2}}}{(\lambda+\mathcal{I}(x))}\left((\lambda+\mathcal{I}(x))\,H_1(x)+\mathcal{I}'(x)H_0(x)\right)\\
&= \sqrt{\frac{1}{2\sqrt{\pi}}}\frac{e^{-\frac{x^2}{2}}}{(\lambda+\mathcal{I}(x))}\left((\lambda+\mathcal{I}(x))\,2x+\mathcal{I}'(x)\right)\\
&= \sqrt{\frac{1}{2\sqrt{\pi}}}\frac{e^{-\frac{x^2}{2}}}{(\lambda+0.5+0.5Erf[x])}\\
&\quad\times\left((\lambda+0.5+0.5Erf[x])\,2x+\frac{e^{-x^2}}{\sqrt{\pi}}\right)\widehat{\psi}_2^-(x,\lambda)\\
\widehat{\psi}_2^-(x,\lambda) &= \sqrt{\frac{1}{8\sqrt{\pi}}}\frac{e^{-\frac{x^2}{2}}}{(\lambda+\mathcal{I}(x))}\left((\lambda+\mathcal{I}(x))\,H_2(x)+\mathcal{I}'(x)H_1(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{1}{8\sqrt{\pi}}}\frac{e^{-\frac{x^2}{2}}}{(\lambda+\mathcal{I}(x))}\left((\lambda+\mathcal{I}(x))\left(4x^2-2\right)+\mathcal{I}'(x)2x\right)\\
&= \sqrt{\frac{1}{8\sqrt{\pi}}}\frac{e^{-\frac{x^2}{2}}}{(\lambda+0.5+0.5Erf[x])}\\
&\quad\times\left((\lambda+0.5+0.5Erf[x])\left(4x^2-2\right)+\frac{e^{-x^2}}{\sqrt{\pi}}2x\right)
\end{aligned}
\tag{24}
$$

3.2 The symmetric Rosen-Morse potential

Let us consider a symmetric Rosen-Morse potential sometimes also called Poschl-Teller potential which is characterized by the superpotential

$$
W(x) = \gamma \tanh x \tag{25}
$$

The corresponding partner potentials are

$$
V_{\pm}(x) = \gamma^2 - \gamma(\gamma \mp 1) sech^2 x \tag{26}
$$

The eigenvalue and the eigen functions for the potential $V_{-}(x)$ are

$$
\begin{aligned}
E_n^- &= \gamma^2 - (\gamma - n)^2, \quad n = 0, 1, 2, ... < [\gamma]\\
\psi_n^-(x,\gamma) &= N_n^-(\gamma) sech^{\gamma-n} x\, P_n(\gamma-n, \gamma-n, \tanh x)
\end{aligned}
\tag{27}
$$

where $N_n^-(\gamma)$ is the normalization constant and $P_n(\gamma-n, \gamma-n, \tanh x)$ is the nth order Jacobi Polynomial [22]. The eigen values and the eigen functions [9] for the partner potential $V_{+}(x)$ are

$$
\begin{aligned}
E_n^+ &= E_{n+1}^-\\
\psi_n^+(x,\gamma) &= \frac{1}{\sqrt{E_{n+1}^-}}\left(\frac{d}{dx} + W(x)\right)\psi_{n+1}^-(x,\gamma) = \psi_n^-(x,\gamma-1)
\end{aligned}
\tag{28}
$$

The eigenvalues and the eigenfunctions for the isospectral potential $\widehat{V}_{-}(x,\lambda)$ are

$$
\begin{aligned}
\widehat{E}_0^- &= 0\\
\widehat{\psi}_0^-(x,\gamma,\lambda) &= \frac{\sqrt{\lambda(\lambda+1)} sech^{\gamma} x}{\sqrt{B(\gamma, 0.5)}(\lambda+\mathcal{I}(x))}\\
\widehat{\psi}_{n+1}^-(x,\gamma,\lambda) &= \psi_{n+1}^-(x,\gamma) + \frac{1}{E_{n+1}^-}\left(\frac{\mathcal{I}'(x)}{\lambda+\mathcal{I}(x)}\right)\left(\frac{d}{dx}+\gamma\tanh x\right)\psi_{n+1}^-(x,\gamma)\\
&= \frac{1}{\lambda+\mathcal{I}(x)}\left((\lambda+\mathcal{I}(x))\,\psi_{n+1}^-(x,\gamma) + \frac{\mathcal{I}'(x)}{\sqrt{E_{n+1}^-}}\psi_n^-(x,\gamma-1)\right)
\end{aligned}
$$

$$= \frac{sech^{\gamma-n-1}(x)}{\lambda+\mathcal{I}(x)}\left(\lambda+\mathcal{I}(x)\right) P_{n+1}(\gamma-n-1,\gamma-n-1,\tanh x)$$
$$+\frac{\mathcal{I}'(x)}{\sqrt{E^{-}_{n+1}}}P_n(\gamma-n-1,\gamma-n-1,\tanh x) \tag{29}$$

where

$$\mathcal{I}(x)=\frac{1}{B(\gamma,0.5)}\int_{-\infty}^{x} sech^{2\gamma}t\,dt \tag{30}$$

For computation of the various lengths it is necessary choose γ. We choose $\gamma = 4$ so that there are three bound states. The energy and the corresponding wave functions for these states can be found from (29) and are given by

$$\begin{aligned}
\widehat{\psi}_0^-(x,\lambda) &= \frac{\sqrt{\lambda(\lambda+1)}sech^{\gamma}x}{\sqrt{B(\gamma,0.5)}(\lambda+\mathcal{I}(x))}\\
&= \frac{\sqrt{1120\lambda(\lambda+1)}sech^4 x}{32\lambda+16+\tanh x(16+8\sec h^2x+6\sec h^4x+5\sec h^6x)}\\
\widehat{\psi}_1^-(x,\lambda) &= \frac{sech^3(x)}{\lambda+\mathcal{I}(x)}\left((\lambda+\mathcal{I}(x))\,P_1(3,3,\tanh x)+\frac{\mathcal{I}'(x)}{\sqrt{7}}P_0(3,3,\tanh x)\right)\\
&= \frac{sech^3(x)}{\lambda+\mathcal{I}(x)}\left((\lambda+\mathcal{I}(x))\,4\tanh x+\frac{\mathcal{I}'(x)}{\sqrt{7}}\right)\\
&= \frac{sech^3(x)}{\lambda+\mathcal{I}(x)}\left((\lambda+\mathcal{I}(x))\,4\tanh x+\frac{35sech^8x}{32\sqrt{7}}\right)\\
\widehat{\psi}_2^-(x,\lambda) &= \frac{sech^2(x)}{\lambda+\mathcal{I}(x)}\left((\lambda+\mathcal{I}(x))\,P_2(2,2,\tanh x)+\frac{\mathcal{I}'(x)}{\sqrt{12}}P_1(2,2,\tanh x)\right)\\
&= \frac{sech^2(x)}{\lambda+\mathcal{I}(x)}\left((\lambda+\mathcal{I}(x))\,(7\tanh^2x-1)+\frac{\mathcal{I}'(x)}{\sqrt{12}}3\tanh x\right)\\
&= \frac{sech^2(x)}{\lambda+\mathcal{I}(x)}\left((\lambda+\mathcal{I}(x))\,(7\tanh^2x-1)+\frac{35sech^8x}{32\sqrt{12}}3\tanh x\right)
\end{aligned} \tag{31}$$

where

$$\mathcal{I}(x)=\frac{1}{32}\left\{16+\tanh x\left(16+8\sec h^2x+6\sec h^4x+5\sec h^6x\right)\right\} \tag{32}$$

4 Entropy length of the linear Harmonic oscillator and its isospectral partner

The position space Fisher length $(\widehat{\delta x}_H)_n^-(\lambda)$ for the state $\widehat{\psi}_n^-(x,\lambda)$ in Eq. (22) is calculated for $n = 0, 1, 2$ and it is a function of λ. In Fig. 1 we plot the Fisher lengths $(\widehat{\delta x}_H)_n^-(\lambda)$ against λ for $\mu = 1$ and $n = 0,\ 1,\ 2$. From Fig. 1 it is seen that the Fisher

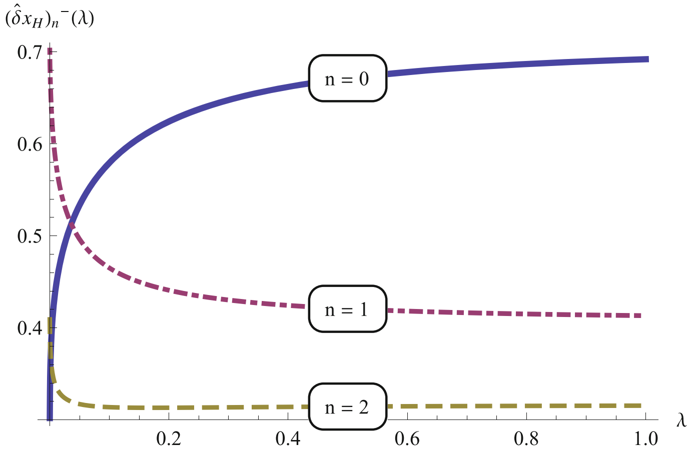

Fig. 1 Fisher lengths of the state $\widehat{\psi}_n^-(x, \mu, \lambda)$ in Eq. (24) in terms of λ for $\mu = 1$

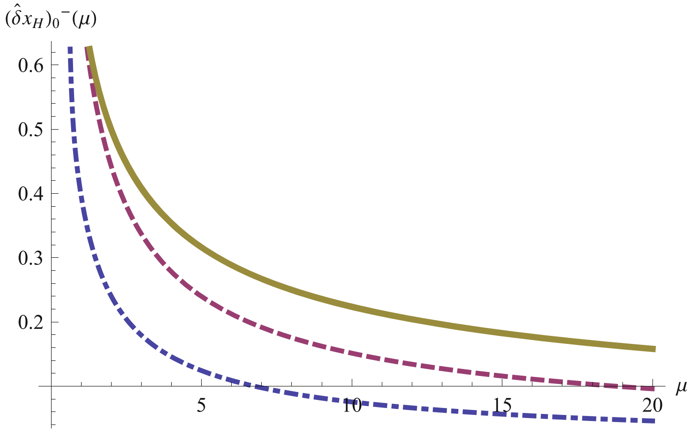

Fig. 2 Fisher lengths of the ground state $\widehat{\psi}_0^-(x, \mu, \lambda)$ in Eq. (22) in terms of μ for $\lambda = 0.0001$ represent *DotDashed curve*, $\lambda = 0.1$ represent *Dashed curve* and $\lambda = 1{,}000$ represent *continuous curve*

length increases as λ increases for $n = 0$ and decreases as λ increases for $n = 1,\ 2$. Also one may note that for all values of λ the Fisher length is an increasing function of the quantum number n.

We now compute the Fisher length $(\widehat{\delta x}_H)_0^-(\mu)$ of the ground state $\widehat{\psi}_0^-(x, \mu, \lambda)$ as a function of μ for different values of λ and the results are presented in Fig. 2. From Fig. 2 we conclude that for a given λ the Fisher length $(\widehat{\delta x}_H)_0^-(\mu)$ decreases as μ increases. Also we may say that the Fisher length $(\widehat{\delta x}_H)_0^-(\mu)$ increases and tends to $(\delta x_H)_0^-(\mu)$ if λ increases to a large value. So we conclude that for the excited states Fisher length for the *orthogonal functions* is less than the corresponding classical orthogonal polynomial which in this case is the Hermite polynomials.

Similarly we calculate the Renyi lengths $\widehat{\mathfrak{L}}_{Hn}^{-R}(\lambda)$ of the state $\widehat{\psi}_n^-(x, \lambda)$ in Eq. (22) for $n = 0, 1, 2$ and it is a function of (λ, q). In Fig. 3 we plot the Renyi length $\widehat{\mathfrak{L}}_{Hn}^{-R}(\lambda)$

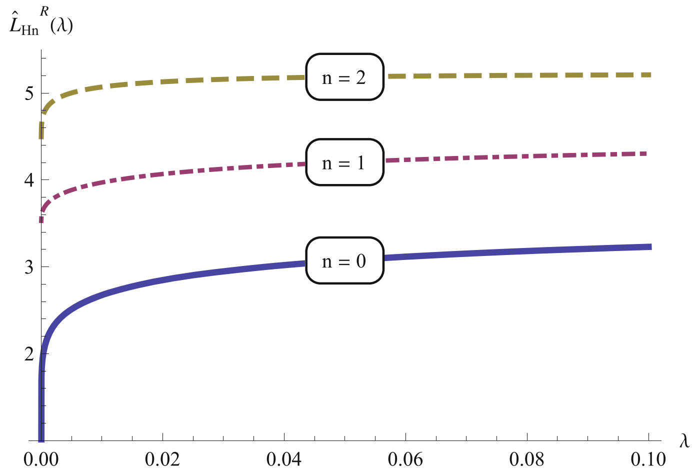

Fig. 3 Renyi lengths of the state $\widehat{\psi}_n^-(x, \mu, \lambda)$ in Eq. (24) in terms of λ for $\mu = 1$

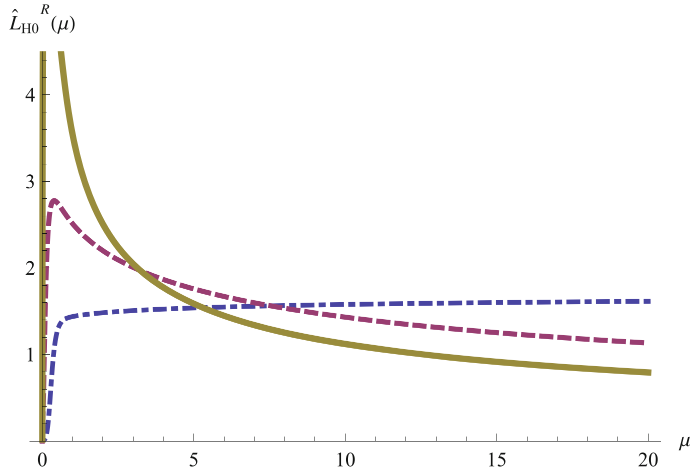

Fig. 4 Renyi lengths of the ground state $\widehat{\psi}_0^-(x, \mu, \lambda)$ in Eq. (22) in terms of μ for $\lambda = 0.01$ represent *DotDashed curve*, $\lambda = 1$ represent *Dashed curve* and $\lambda = 1{,}000$ represent *continuous curve*

as a function of λ for $\mu = 1, q = 0.5$ and $n = 0, 1, 2$. From Fig. 3 we can say that as a function of λ the Renyi length increases as λ increases for any values of the quantum number n. Also it increases as the quantum number n increases for any value of λ.

In Fig. 4 we plot the Renyi-length of the ground state $\widehat{\psi}_0^-(x, \mu, \lambda)$ as a function of μ for $q = 0.5$ and different values of λ. From Fig. 4 the Renyi lengths $\widehat{\mathfrak{L}}_{H0}^{-R}(\mu, \lambda)$ are found to decrease as μ increases for large values of λ. Furthermore the Renyi-length $\widehat{\mathfrak{L}}_{H0}^{-R}(\mu, \lambda)$ decreases and tends to the Renyi lengths $\mathfrak{L}_{H0}^{-R}(\mu)$ as λ becomes large for $q < 1$ and the Renyi-length $\widehat{\mathfrak{L}}_{H0}^{-R}(\mu, \lambda)$ tends to the Renyi lengths $\mathfrak{L}_{H0}^{-R}(\mu)$ if λ increases and $q > 1$. So we conclude that the Renyi length of the orthogonal functions

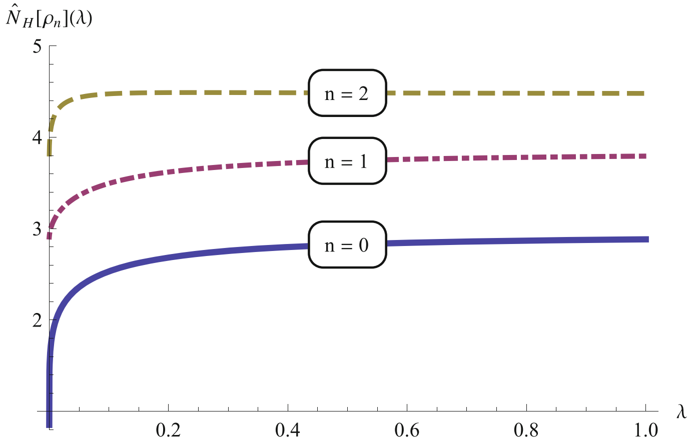

Fig. 5 Shannon lengths of the state $\widehat{\psi}_n^-(x, \mu, \lambda)$ in Eq. (24) in terms of λ for $\mu = 1$

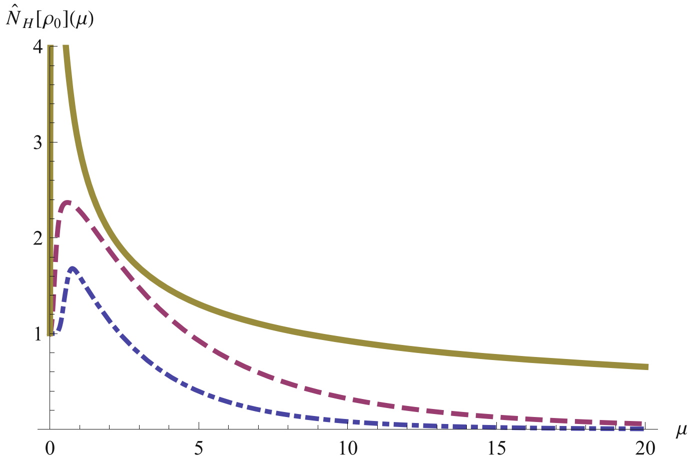

Fig. 6 Shannon lengths of the ground state $\widehat{\psi}_0^-(x, \mu, \lambda)$ in Eq. (22) in terms of μ for $\lambda = 0.0001$ represent *DotDashed curve*, $\lambda = 0.1$ represent *Dashed curve* and $\lambda = 1{,}000$ represent *continuous curve*

are greater than or less than that of the corresponding orthogonal polynomial according as $q < 1$ or $q > 1$.

The Shannon length $\widehat{N}_H[\rho_n](\lambda)$ of the same state $\widehat{\psi}_n^-(x, \lambda)$ in Eq. (22) is a function of (λ). In Fig. 5 we plot the Shannon lengths $\widehat{N}_H[\rho_n](\lambda)$ as a function of λ for $\mu = 1$. From Fig. 5 we infer that the Shannon length increases as λ increases for any values of the quantum number n. Also it increases as the quantum number n increases for all values of λ.

Next in Fig. 6 we plot the Shannon length of the ground state $\widehat{\psi}_0^-(x, \mu, \lambda)$ as a function of μ for different values of λ. From Fig. 6 we can say that the the Shannon length $\widehat{N}_H^-[\rho_0](\lambda)$ decreases as μ increases for any fixed value of λ. The Shannon length $\widehat{N}_H^-[\rho_0](\lambda)$ increases and approaches the Shannon lengths $N_H^-[\rho_0](\lambda)$ as λ

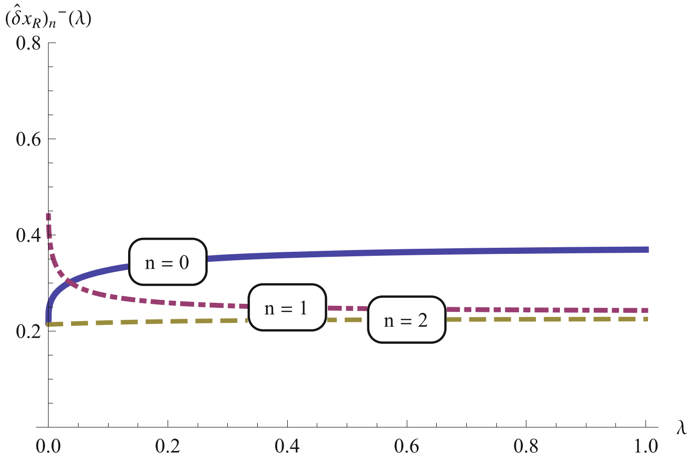

Fig. 7 Fisher lengths of the state $\widehat{\psi}_n^-(x, \gamma, \lambda)$ in Eq. (31) in terms of λ for $\gamma = 4$

increases. So we conclude that the Shannon length of the orthogonal functions are less than that of the corresponding polynomial.

5 Entropy length of the symmetric Rosen-Morse potential and the corresponding isospectral partner potentials

The position space Fisher length $(\widehat{\delta x_R})_n^-(\lambda)$ for the state $\widehat{\psi}_n^-(x, \lambda)$ in Eq. (29) is calculated for $n = 0, 1, 2$ and it is a function of λ. Now we plot the Fisher lengths $(\widehat{\delta x_R})_n^-(\lambda)$ as a function of $\lambda$and $\gamma = 4$, $n = 0, 1, 2$ in Fig. 7. From Fig. 7 it is seen that the Fisher length increases as λ increases for $n = 0$ and it decreases as λ increases for $n = 1, 2$. Also it can be seen that for any fixed value of λ the Fisher length increases with the quantum number n.

Next we plot the Fisher length $(\widehat{\delta x_R})_0^-(\gamma)$ of the ground state $\widehat{\psi}_0^-(x, \gamma, \lambda)$ as a function of γ for different values of λ in Fig. 8. From Fig. 8 we conclude that the Fisher length $(\widehat{\delta x_R})_0^-(\gamma)$ decreases as γ increases when λ is kept fixed. Also the Fisher length $(\widehat{\delta x_R})_0^-(\gamma)$ increases to the Fisher length $(\widehat{\delta x_R})_0^-(\gamma)$ if λ incrreases. So we conclude that Fisher length of the orthogonal functions are less than that of the corresponding polynomial which in this case is the Jacobi polynomial.

Similarly we calculate the Renyi lengths $\widehat{\mathfrak{L}}_{Rn}^{-R}(\lambda)$ of the state $\widehat{\psi}_n^-(x, \lambda)$ in Eq. (29) for $n = 0, 1, 2$ and it is a function of (λ, q). Now we plot the Renyi length $\widehat{\mathfrak{L}}_{Rn}^{-R}(\lambda)$ as a function of λ in Fig. 9 for $\gamma = 4, q = 0.5$ and $n = 0, 1, 2$. From Fig. 9 we can say that the Renyi length increases as λ increases for any values of the quantum number n. Also it is increases as the quantum number n increases for a fixed values of λ as well as for any values of λ.

Also plot the Renyi-length of the ground state $\widehat{\psi}_0^-(x, \gamma, \lambda)$ as a function of γ for $q = 0.5$ and different values of λ in Fig. 10. From Fig. 10 the Renyi lengths $\widehat{\mathfrak{L}}_{R0}^{-R}(\lambda)$ decreases as γ increases for a large values of λ. The Renyi-length $\widehat{\mathfrak{L}}_{R0}^{-R}(\mu, \lambda)$ increases

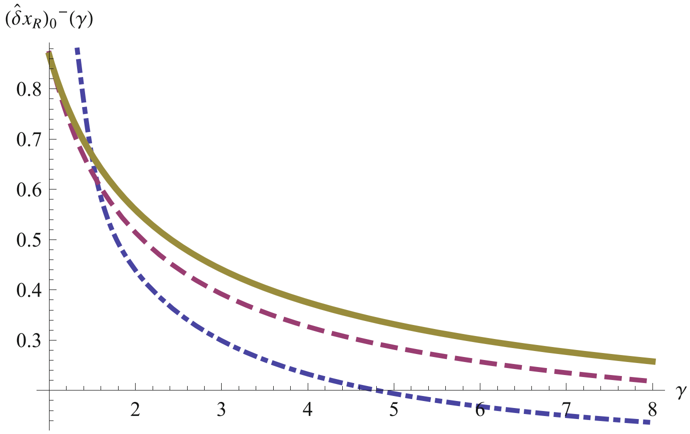

Fig. 8 Fisher lengths of the ground state $\widehat{\psi}_0^-(x,\gamma,\lambda)$ in Eq. (29) in terms of γ for $\lambda = 0.0001$ represent *DotDashed curve*, $\lambda = 0.1$ represent *Dashed curve* and $\lambda = 1{,}000$ represent *continuous curve*

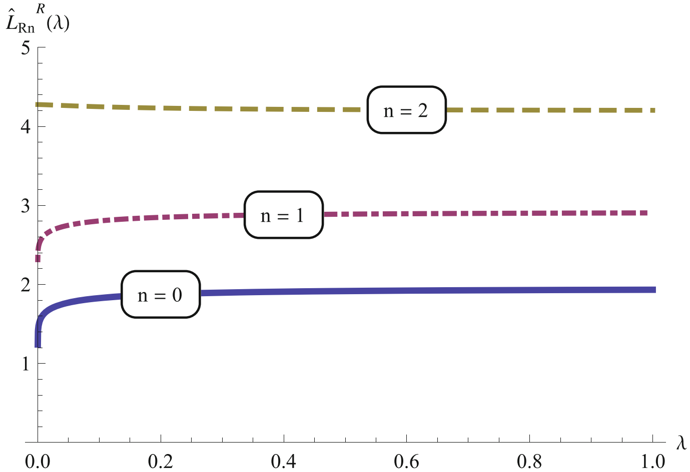

Fig. 9 Renyi lengths of the state $\widehat{\psi}_n^-(x,\gamma,\lambda)$ in Eq. (31) in terms of λ for $\gamma = 4$

and increased to the Renyi lengths $\mathfrak{L}_{R0}^{-R}(\mu)$ if λ increases and increased to a large number for $q = 0.5$. So we conclude that Renyi length of the orthogonal functions are less than that of the corresponding polynomial.

The Shannon length $\widehat{N}_R[\rho_n](\lambda)$ of the same states $\widehat{\psi}_n^-(x,\lambda)$ in Eq. (29) is a function of (λ). Now we plot the Shannon lengths $\widehat{N}_R[\rho_n](\lambda)$ as a function of λ in Fig. 11 for $\gamma = 4$. From Fig. 11 we can say that the Shannon length slowly increases as λ increases for any values of the quantum number n and it is also increases as the quantum number n increases for a fixed values of λ as well as any values of λ.

Also plot the Shannon length of the ground state $\widehat{\psi}_0^-(x,\gamma,\lambda)$ as it is a function of γ for different values of λ in Fig. 12. From Fig. 12 we can say that the the Shannon length $\widehat{N}_R^-[\rho_0](\lambda)$ is decreases as γ increases for a fixed values of λ. The Shannon

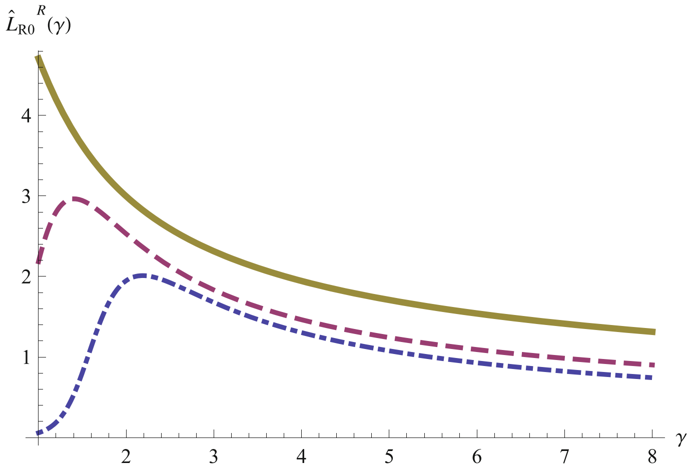

Fig. 10 Renyi lengths of the ground state $\widehat{\psi}_0^-(x, \gamma, \lambda)$ in Eq. (29) in terms of γ for $\lambda = 0.00001$ represent *DotDashed curve*, $\lambda = 0.001$ represent *Dashed curve* and $\lambda = 1{,}000$ represent *continuous curve*

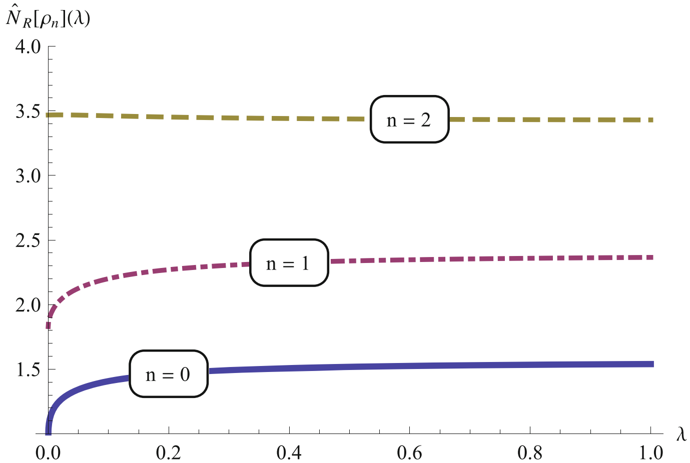

Fig. 11 Shannon lengths of the state $\widehat{\psi}_n^-(x, \gamma, \lambda)$ in Eq. (31) in terms of λ for $\gamma = 4$

length $\widehat{N}_R^-[\rho_0](\lambda)$ increases and increased to the Shannon lengths $N_R^-[\rho_0](\lambda)$ as λ increases and increased to a large number. So we conclude that the Shannon length of the orthogonal functions are less than that of the corresponding polynomial.

6 Observation

Here we summarize the findings of our computation.

For the Harmonic oscillator

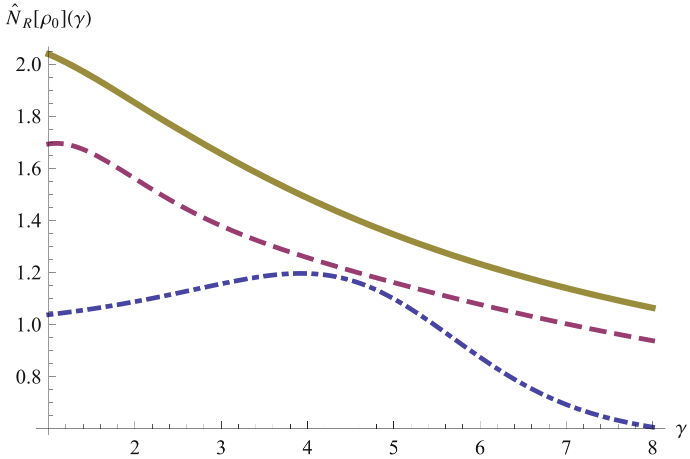

Fig. 12 Shannon lengths of the ground state $\widehat{\psi}_0^-(x, \gamma, \lambda)$ in Eq. (29) in terms of γ for $\lambda = 0.001$ represent *DotDashed curve*, $\lambda = 0.1$ represent *Dashed curve* and $\lambda = 1,000$ represent *continuous curve*

- Fisher lengths of the orthogonal functions are less than that of the corresponding associated orthogonal polynomial for any quantum number n and for any value of μ.
- Renyi lengths of the orthogonal functions are less than or greater than that of the corresponding associated orthogonal polynomials according as $q > 1$ or $q < 1$ for any quantum number n and for any value of μ.
- Shannon lengths of the orthogonal functions are less than that of the corresponding associated orthogonal polynomials for any quantum number n and for any values of μ.

For the Rosen-Morse potential

- Fisher lengths of the orthogonal functions are less than that of the corresponding associated orthogonal polynomials for any quantum number n and $\gamma = 4$.
- Renyi lengths of the orthogonal functions are less than that of the corresponding associated orthogonal polynomials for any quantum number n; $q = 0.5$ and $\gamma = 4$.
- Shannon lengths of the orthogonal functions are less than that of the corresponding associated orthogonal polynomials for any quantum number n and $\gamma = 4$.

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